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COMPLETELY PRIMARY ALGEBRAS WHICH ARE

DIRECT PRODUCTS MODULO THE RADICAL

by

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I. INTRODUCTION

A. Cohomological Background for the Problem

Cohomology theory for groups, rings, and algebras is a recent development (5), aimed at the solution of problems of structure theory which are not amenable to the older methods of investigation. As applied to algebras, cohomology theory has been directed to the problem of decomposing a module into the direct sum of invariant sub-modules (7, 8, 9). A consequence of this approach has been the clarification of the meaning of the vanishing of all n-dimensional cohomology groups of an algebra (6, 15). This development conspicuously neglects the structure theory of algebras which fail to satisfy Wedderburn's Principal Theorem (1, 27). The investigation of algebras of this type with the aid of cohomology theory is the motivation for this thesis.

B. General Concepts

Some of the definitions which will be used in the succeeding chapters are presented here. These definitions are to be found in Hochschild (8). For a presentation of the subject enveloping these definitions as applied to rings rather than algebras, one may refer to Kawada (13).

Throughout this thesis, R denotes at most a semi-simple associative algebra of finite order, with an identity, over a field K, while A denotes an algebra of finite order, with an identity, over K, created from R and an R-R module in a manner to be described. A module P is an R-R module if P is an R-right and an R-left module, satisfies $(ap)b = a(pb)$ for all a and b in R and any p in P, and is such that the identity, 1, of R is both the left and right identity operator on P. The following are the definitions of concepts frequently encountered in this thesis:

Definition 1. If A contains a 2-sided ideal P such that $A/P \cong R$, and such that P is an R-R module on which R and A/P coincide modulo P^2, then A is called an extension* of P by R.

Definition 2. The extension, A of P by R, is called

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*Given any associative algebra A which has R as a homomorphic image, one may always regard the kernel P in the manner just indicated, modulo P^2.

singular (whether or not R is semi-simple) if and only if $P^2 = 0$. Of course, every extension, when considered modulo P^2, is singular. Since our R is semi-simple, then P is the radical N of A whenever A is an associative extension.

Definition 3. An m-dimensional P-cochain of R is an m-linear (homogeneity always implied) map g of R into P. The set of all m-dimensional P-cochains of R is a vector space over K called $C^m(R,P)$.

Definition 4. The coboundary operator δ, which acts on an (m-1)-dimensional P-cochain g of R is defined by:

$$(\delta g)(x_1,---,x_m) = x_1 g(x_2,---,x_m) + (-1)^m g(x_1,---,x_{m-1})x_m + \sum_{n=1}^{m-1}(-1)^n g(x_1,---,x_n x_{n+1},---,x_m)$$

for every set of elements $(x_1,---,x_m)$ of R. It is easily shown that $\delta^2 g = \delta(\delta g) = 0$.

Definition 5. An m-dimensional P-cochain g of R is called an m-dimensional P-cocycle of R if and only if $(\delta g)(x_1,---,x_m)$ vanishes for all sets $(x_1,---,x_m)$ in R. The set of all m-dimensional P-cocycles of R forms a vector space over K denoted by $Z^m(R,P)$.

Definition 6. An m-dimensional P-cocycle g of R is said to be a cobounding cocycle if and only if there exists an $(m-1)$-dimensional P-cochain f of R such that $g = \delta f$. The set of all m-dimensional cobounding P-cocycles of R is a vector space over K denoted by $B^m(R,P)$.

Definition 7. The m-dimensional cohomology group of R for P, denoted by $H^m(R,P)$, is the vector space $Z^m(R,P)/B^m(R,P)$.

Thus, the set $\{g_i\}$ of all m-dimensional P-cocycles of R can be divided into cohomology classes, that is, classes* of cocycles which differ only by a coboundary.

Definition 8. Two singular extensions A and A' of P by R are said to be isomorphic if there exists an isomorphism I of A onto A' such that $s'I = s$, where s and s' are the homomorphisms of A and A' into R, respectively.

It is appropriate to introduce here a theorem proved by Hochschild (7, p. 67), which is basic to this thesis: "There is a one to one correspondence between classes of isomorphic singular extensions of R and the 2-dimensional cohomology classes of R." It follows in particular that to all the

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*Although g in $H^n(R,P)$ signifies that g is a cohomology class, it is convenient to use this notation to mean that g is a non-cobounding cocycle in $Z^n(R,P)$.

elements of $B^2(R,P)$ there corresponds one singular extension of P by R, while to each class of non-cobounding 2-cocycles there corresponds one singular extension of P by R.

C. Properties of Non-cobounding 2-cocycles

Let A be an extension of P by R. By the isomorphism $A/P \cong R$ there exists a linear and homogeneous correspondence $r \leftrightarrow A_r$ for any r in R, where A_r is a representative of the residue class to which r corresponds in $A/P \cong R$. Define the function g for x and y in R by $g(x,y) = A_x A_y - A_{xy}$. From the homomorphism it follows that the values of $g(x,y)$ are in P. Furthermore, g belongs to $C^2(R,P)$. For, if h and k are in K, $g(hx + ky,z) = A_{hx + ky}A_z - A_{(hx + ky)z}$ or $g(hx + ky,z) = hg(x,z) + kg(y,z)$, while similarly the linearity of g for the second argument can be shown.

The set P_0 generated over A by $g(x,y)$ for x and y in R is an R-R sub-module of the R-R module P because the definition of singular extension implies in particular that $xg(y,z) = A_x g(y,z)$ and $g(y,z)x = g(y,z)A_x$ for all x, y, and z in R.

Lemma 1. (7) Let A be a singular extension of P by R. Then $g'(x,y) = g(x,y) + (\delta f)(x,y)$ for a one-dimensional P-cochain f of R if and only if $g'(x,y) = A'_x A'_y - A'_{xy}$ where $A'_x - A_x = f(x)$ in P.

Proof: Let $g'(x,y) = A'_x A'_y - A'_{xy}$ be a new cochain obtained by selecting a new linear set of representatives of the classes of A/P. Then since $A'_x = A_x + f(x)$, for some $f(x)$ in P, $g'(x,y)$ can be written $g'(x,y) = (A_x + f(x))(A_y + f(y)) - (A_{xy} + f(xy))$ or $g'(x,y) = g(x,y) + (\delta f)(x,y)$.

Conversely, let $g'(x,y) = g(x,y) + (\delta f)(x,y)$. Then using the definition of $g(x,y)$, $g'(x,y)$ can be written

$$\begin{aligned} g'(x,y) &= A_x A_y - A_{xy} + xf(y) - f(xy) + f(x)y \\ &= A_x A_y - A_{xy} + A_x f(y) - f(xy) + f(x)A_y \\ &= A'_x A'_y - A'_{xy} \end{aligned}$$

where A'_x is a new choice of representatives of the class $(x + P)$.

Regardless of the choice of a linear representative set for A/P, there is then only one 2-dimensional P-cochain of $H^2(R,P)$ associated with A. Under the assumption that A is associative, it is true that this cochain is a cocycle. In fact

Lemma 2. (13) If A is associative, then g is a cocycle.

Furthermore, A is an associative singular extension if and only if g is a cocycle.

Proof: $(\delta g)(x,y,z) = A_x(A_yA_z - A_{yz}) - (A_{xy}A_z - A_{xyz})$
$+ (A_xA_{yz} - A_{xyz}) - (A_xA_y - A_{xy})A_z$
$= A_x(A_yA_z) - (A_xA_y)A_z.$

Hence, if A is associative, this difference vanishes for any x, y, and z in R. For the converse, suppose g is a cocycle, so that the representatives $\{A_x\}$ are associative. There then remains to be shown that $(pA_x)A_y = p(A_xA_y)$ and $(A_xp)A_y = A_x(pA_y)$ for any p in P. But $(pA_x)A_y = (pA_x)y = (px)y = p(xy) = p(A_{xy}) = p(A_xA_y - P') = p(A_xA_y)$ for some P' in P. In a similar manner the second equality can be demonstrated. Henceforth, A is an associative algebra.

Definition 9. A = ext(R,P,g) is the extension of P by R using the cocycle g.

The sub-module of P generated by g(x,y), which has been called P_0, need not be all of P. Since in our case P is the radical N of A, let P and P_0 be denoted by N and N_0, respectively. Then $N_0 = Rg(R,R)R$, that is, if n_0 is in N_0, then $n_0 = \sum r^1 g(r^2,r^3)r^4$ for r^i in R. The maximum sub-algebra of A which has N_0 as radical and g as cocycle will be denoted by A_0. Then it is noted that $A_0 = \mathrm{ext}(R,N_0,g)$, that is, A_0 is an extension of N_0 by R, where $N_0 = Rg(R,R)R$ and $A_0/N_0 \cong A/N$. A useful fact about the cocycle g is that it can be normalized over K.

Definition 10. g is a normalized cocycle with respect to K if g(x,y) vanishes for all values of x or y in K.

Lemma 3. If $A = \mathrm{ext}(R,N,g)$ is (not necessarily a singular) extension of R, there exists a 2-dimensional cocycle g' in $Z^2(R,N)$ such that $\mathrm{ext}(R,N,g) = \mathrm{ext}(R,N,g')$ and such that $g'(k,x) = g'(x,k) = 0$ for any x in R, and any k in K.

Proof: Let $\{A_x'\}$ be a new set of representatives of A/N, defined by $A_x' = A_x$ for all $x \neq 1$ in R, and $A_1' = e$, where e is the identity of A. Under the isomorphism $R \cong A/N$, if 1 is the identity of R, then A_1 is in the class $e + N$. Hence, $A_1' = e$ is really a representative of the same class that A_1 was. Since the other representatives are left unaltered, the set $\{A_x'\}$ is actually a new choice of representatives of A/N. Furthermore, $g'(1,y) = eA_y' - A_y' = 0$ and $g'(x,1) = A_x'e - A_x' = 0$ for all x and y in R. Then due to the homogeneity of g, it follows that $g(x,k) = g(k,x) = 0$ for all k in K.

Lemma 4. If g is a cobounding cocycle, the normality of g implies the normality of the 1-cochain f such that $g = \delta f$.

Proof: If $g(x,y) = xf(y) - f(xy) + f(x)y$, then $0 = g(1,y) = f(y) - f(y) + f(1)y$. Consequently, $f(1) = 0$. Since f is a cochain, then f is homogeneous, and $f(k) = 0$ for all k in K.

The remainder of this thesis will be largely concerned with non-cobounding cocycles g of R into N. The normalized cocycle g' produces the same extension of N by R as the non-normalized g. It is, therefore, of no advantage to consider non-normalized N-cocycles, g of R, and henceforth g is a normalized cocycle with respect to K. It is emphasized, in view of the preceding development, that the following assumptions may now be permanently adopted: Whenever A is a singular extension by R, then A and R are associative algebras, with identities, of finite order over K; R is semi-simple; and g is the normalized N-cocycle of R for this extension.

D. Cleavage and Cohomology

Singular extensions with non-cobounding g's are intimately related to the concept of cleavage (22).

Definition 11. An algebra A over K is cleft with respect to K if it is associative and contains a sub-algebra A^* over K such that $A/N \cong A^*$, and $A = N + A^*$ as a group direct sum. Otherwise, A is uncleft.

Hochschild (7, 8, 9) introduced the notion of segregation. An algebra R is said to be segregated in an extension A of P by R if A contains a module P and a sub-algebra A^* such that $A/P \cong R$ and $A = P + A^*$, as an R-R module direct sum. When R is semi-simple this is related to the study of the cleavage of A. Instead of studying the extension modules of P by R, cleavage studies the structure of a given algebra A to determine whether or not, or to what degree, the algebra can be written as a direct sum, $A^* + N$. The classical result on cleavage is Wedderburn's theorem for algebras of characteristic zero (27). For this thesis, it is the case of non-zero characteristic which is of major concern.

Theorem 1. Every uncleft algebra A contains a subalgebra $A^* = A^*(g)$ with radical $N^* = N^*(g)$ such that A^* is generated by every linear residue system of A^*/N^*, and such that $A^*/N^* = A/N$. Furthermore, g is non-cobounding.

Proof: Define as before $N_0 = Rg(R,R)R$ and $A_0 = \text{ext}(R,N_0,g)$. If A_x' is any different linear set of representatives for A/N, such that $A_x' = A_x + n_0(x)$ for $n_0(x)$ in N_0, take $g'(x,y) = A_x'A_y' - A_{xy}'$. It follows that $N_0' = Rg'(R,R)R$ is a subalgebra of N_0. Choose g', if possible, such that the dimension of N_0' is less than or equal to the dimension of N_0. Due to the finiteness of dimension of A, the continuation of this process will ultimately replace N_0 by the zero ideal or by an unshrinkable non-zero nilpotent ideal N^*. The first case is not possible since A is uncleft. There is, then, a minimum nilpotent ideal N^* contained in N_0, where $N^* = Rg^*(R,R)R$, g^* being defined by $g^*(x,y) = A_x^*A_y^* - A_{xy}^*$. Here $\{A_x^*\}$ is a new choice of representatives of A/N, having been chosen by adding an element of the last reduction of N_0. Since all of N^* is generated by R acting on $g^*(x,y)$ right and left, and similarly by all alternative 2-cocycles, $A^* = \text{ext}(R,N^*,g^*)$ is generated by every linear residue system of A^*/N^*. For the final statement in the theorem, suppose $g(x,y) = (\delta f)(x,y)$. Now choose $A'_x = A_x - f(x)$. Then $g'(x,y) = A'_x A'_y - A'_{xy} = 0$. Hence A is cleft, in contradiction to the assumption of the theorem.

The sub-algebra A^* of this theorem is an example of the absolute opposite of a cleft algebra. The property illustrated by A^* is involved in the central theme of this thesis, and is called maximal uncleavage. This idea was introduced by Vinograde (24), and is defined as follows:

Definition 11. Let (T) be the algebra generated by an arbitrary set of representatives T, that is, a residue system, for A/N. Then A is maximally uncleft (briefly, m.u.) if and only if (T) = A for every residue system T.

That the m.u. property is really the extreme opposite of cleavage has been demonstrated by Vinograde (24), who shows in particular that if an algebra A is m.u., A/I is uncleft for every ideal I properly contained in N, and conversely. It is true however that the property of cleavage and the associated properties of uncleavage and maximal uncleavage depend upon what is considered to be the base field. This will be forcefully brought out in the last section of this paper, on field-composites. That is, an algebra can cleave for one base field, and be m.u. for another base field. Since the algebras ultimately studied in this thesis are m.u. algebras, we will always take $N^* = Rg(R,R)R$ for our radical N, and so $A^* = A$, unless otherwise explicitly indicated, as in the second chapter.

The relationship between the non-cobounding cocycle g, and maximal uncleavage can be further emphasized by considering the lemmas below, given here without proof. The first is due to Vinograde (26), while the second is due to Hochschild (7).

Lemma 5. An associative algebra A is m.u. if and only if A/N^2 is m.u.

Lemma 6. R is segregated in every extension A (that is, every A is cleft) if and only if it is segregated in every singular extension, and the latter holds if and only if $H^2(A,P) = 0$ for all P.

In comparing this result with Lemma 5, one must note that every R has at least one non-trivial cleft extension, so that no strict analogy to Lemma 6 can be expected for the m.u. property. But Lemma 5 shows that the structure of algebras with $N^2 = 0$ is the key to the study of m.u. algebras. Consequently, in this thesis, A is a singular algebra, that is, an algebra A with a radical N of index two. If the following definition is made, the m.u. property can be further simplified:

Definition 12. A residue system T is called a linear residue system if there exists a subset $\{t_i\}$ of representatives belonging to T, such that the t_i are linearly independent over K and each element t of T can be expressed as a linear combination of the t_i over K. For any residue system T, a maximal subset of linearly independent representatives is called a basis set for T.

Theorem 2. A is m.u. if and only if every linear T generates A.

Proof: Let $\{t_i\}$ be a basis set of representatives for an arbitrary T. Let T* be the linear residue set generated by $\{t_i\}$. If A is m.u., then A is generated by every residue set T, regardless of the choice of representatives of T, hence by T*. Conversely, if some (T) $\neq$ A, then a contradiction arises from (T*) = A when T* is constructed from any basis set in T.

E. The Problem

With the foregoing as background, the central problem of this thesis can be formulated. There are two common means of forming "products" of fields over a common base field. These are: 1. The direct product* L X M of two fields L and M over K, and, 2. The field-composite* LM of L and M over K. The direct product of two fields need not, of course, be a field, while the composite of two fields, although a field, need not be unique. A composite of two algebras A and B over K is an algebra C such that C contains algebras A* and B*, where A $\cong$ A* and B $\cong$ B* over K, and C is generated by A and B. It follows that L X M and LM are special composites. The direct product of two algebraic extension

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*See, for example, Albert (1, 2) or Pickert (19).

fields L and M of a common base field K is an algebra L X M containing two sub-fields L* and M* (call them L and M) isomorphic to L and M over K respectively, and such that L X M is equal to $\{\sum a \times b\}$ for a in L and b in M. The order of L X M over K is the product of the orders of L and M over K, with a X b = b X a for all a in L and b in M. The field-composites of L and M are defined as follows: Let Z be a field over K with a sub-field over K equivalent (isomorphic with K fixed) to L and a sub-field over K equivalent to M. The field-composites of L and M with respect to Z are the fields (L_0, M_0) where L_0 and M_0 range over all sub-fields of Z equivalent to L and M. Here (L_0, M_0) is the intersection of all sub-fields of Z which contain L_0 and M_0. Although L X M is not in general a field or even semi-simple, it is a direct sum of composites of L and M when considered modulo its radical. Thus, in L X M, the radical, if not 0, represents the deterioration from semi-simplicity, and in pure inseparable cases, reflects the failure of L X M to be a field.

Any completely primary algebra C (an algebra C with radical N is completely primary if C/N is a field), which modulo its radical is isomorphic to a field composite LM of fields

L and M maybe regarded as a product, say $L^{\circ}M$. However, its radical is not essential to the structure unless C is uncleft, hence, the m.u. case is the natural one to call $L^{\circ}M$. But in view of Lemma 5, the existence and the structure of $L^{\circ}M$ depends upon the existence and structure of singular products, $L^{\circ}M$.

The problem treated in this thesis is that of the existence and structure of various types of singular products, $L^{\circ}M$. The structure of $L^{\circ}M$ is analyzed from the point of view of cohomology, that is, $L^{\circ}M$ is regarded as an extension. The underlying device in the analysis of $L^{\circ}M$ will be to replace its dependence on the cohomology group $H^2(LM,P)$ by $H^2(LM,LM)$, where LM is a field composite. This reduction (the projection theory of the second chapter) is attained at the sacrifice of almost all non-commutativity. Lines of generalization suggest themselves but will not be explored in this thesis.

The efforts of this thesis are centered upon $L^{\circ}M$ when the field composite LM has for factors simple pure inseparable field extensions of the base field K. The reason for restricting L and M to pure inseparable fields is that for any commutative algebra Pickert (17) has shown that the structure theory can be reduced to the study of pure inseparable fields. Though the algebra L^oM is not here assumed commutative, it is deemed adequate by the author to reduce this case too, since primary interest attaches to the commutative case.

Pursuing this end, Chapter II will investigate the so-called projections of g on a field R to produce a cohomological condition for the centrality of the radical of A. Then utilizing this result, Chapter III will develop the relationship between the m.u. property of singular extensions L^oM and the projections of g. Then Chapter IV studies cocycles on simple pure inseparable fields R onto R to establish the criterion that for an algebra A, the projections of g in R are non-cobounding cocycles, and for every set of independent non-cobounding cocycles of R to R there corresponds an m.u. extension A. In the last sections of Chapter IV, it is shown that the values of the projections of g for the m.u. extension A by the direct product of two simple pure inseparable fields are determined by their values on the factors of the direct product. The last section then makes a beginning of the study of $L \circ M$ when $L^{\circ}M = L \times M$ which is not a field, yielding some information about the m.u. property.

II. PROJECTIONS

A. The Projections of g on R

In this chapter, the restrictions on R, which will lead to the study of singular m.u. completely primary algebras L^oM in Chapter IV are introduced gradually.

Definition 13. Let $\{n_i\}$ be any fixed finite subset of N. If $g(x,y) = \sum_i \lambda_i(x,y)n_i$ where $\lambda_i(x,y)$ is in R, for all x and y in R, the set $\{\lambda_i\}$ is the set of projections of g on R with respect to the set $\{n_i\}$.

It was first proved by Hochschild (8) that if R is an inseparable semi-simple algebra over a field K, there exists a non-vanishing 2-cohomology group* of R for some R-module P. This, of course, means that there is some non-cobounding 2-dimensional P-cocycle for R for some P.

If now R is a semi-simple algebra over K, and if V is a finite dimensional R-left module, then V has the minimum

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*For a generalization of this theorem, see (6).

condition over R. If $RV = V$, then according to Artin, Nesbitt, and Thrall (3, p. 36) V is the direct sum of a finite number of its irreducible R-left spaces, say $V = \sum_i V_i$, where V_i is an irreducible R-left sub-space of V. But then if v_i is any non-zero element of V_i, $V_i = Rv_i$ (3, p. 4) for all i. Hence, any $g(x,y)$ in V can be written $g(x,y) = \sum_i \lambda_i(x,y)v_i$ for $\lambda_i(x,y)$ in R. Since N is an R-R module, it is in particular a finite dimensional R-left module over K. Hence, V can be replaced by N throughout this paragraph, and v_i can be denoted by n_i, where n_i is in N. Then $g(x,y) = \sum_i \lambda_i(x,y)n_i$, and $g(x,y) = 0$ if and only if $\lambda_i(x,y)n_i = 0$ for every i.

Lemma 7. If R is a semi-simple algebra over a field K, and $g(x,y) = \sum_i \lambda_i(x,y)n_i$, then $\lambda_i(x,y)n_i$ is an element of $C^2(R,N)$.

Proof: $\lambda_i(x,y)$ is certainly a map of (R,R), that is, the group direct sum, onto R, so that $\lambda_i(x,y)n_i$ is a map of (R,R) onto N. All that remains to be shown is that $\lambda_i(x,y)n_i$ is a bilinear map. Using the fact that g is bilinear, $\sum_i \lambda_i(hx + ky,z)n_i = \sum_i (h\lambda_i(x,z)n_i + k\lambda_i(y,z)n_i)$. Equating the terms which belong to the same irreducible sub-space of N produces $\lambda_i(hx + ky, z)n_i = h\lambda_i(x,z)n_i + k\lambda_i(y,z)n_i$ for all i and all elements x, y, and z in R, and h and k in K. After a symmetric demonstration on the second argument, it follows that $\lambda_i(x,y)n_i$ is an element of $C^2(R,N)$.

The next theorem will give a condition for the $\{\lambda_i(x,y)n_i\}$ to be cocycles.

Theorem 3. Let R be a semi-simple algebra over K, and let A be an algebra with a radical N such that $A/N \cong R$. Let $Rg(R,R)R = N$, where g is a 2-dimensional cocycle of $Z^2(R,N)$. If N is contained in the center of A (that is, if $rn = nr$ for all n in N and r in R) then $\lambda_i n_i$ is in $Z^2(R,N)$ for all i.

Proof: Since g is a 2-cocycle,

(3:1) $\quad 0 = \sum_i (x\lambda_i(y,z)n_i - \lambda_i(xy,z)n_i + \lambda_i(x,yz)n_i - \lambda_i(x,y)n_i z).$

Each term in the ith term of the sum is an element of Rn_i, since $\lambda_i(x,y)n_i z = z\lambda_i(x,y)n_i$. Consequently, the ith term of (3:1) must vanish for each i. Hence $(\delta\lambda_i n_i)(x,y,z) = 0$, and the theorem is proved.

Now, if R is more closely restricted, a direct result can be obtained concerning any set of projections of g on R.

Theorem 4. Let $A = ext(R,N,g)$ be an extension by a division algebra R over a field K, with the dimension of N over R as a left space being s. If $1, a_2, ---, a_t$ is a basis of R over K, then there exists a set of pairs of the basis elements $\{(b_i, c_i)\}$ such that $\{g(b_i, c_i)\}$ is a basis of N over R as a left space.

Proof: Since $N = Rg(R,R)R$, then every element n in N can be written as

$$(4{:}1) \qquad n = (\sum_i k_i^1 a_i) g(\sum_i k_i^2 a_i, \sum_i k_i^3 a_i)(\sum_i k_i^4 a_i)$$

where the k_i^j are in K for $j = 1,2,3,4$. Since the k_i^j are in K (4:1) can be rewritten as

$$(4{:}2) \qquad n = \sum k_i^1 k_j^2 k_k^3 k_l^4 a_i g(a_j, a_k) a_l .$$

Consider now $a_i g(a_j, a_k) a_l$. Since g is a cocycle,

$$(4{:}3) \quad g(a_j, a_k) a_l = a_j g(a_k, a_l) - g(a_j a_k, a_l) + g(a_j, a_k a_l).$$

Going again to equation (4:2), there follows

$$(4{:}4) \quad n = \sum k_i^1 k_j^2 k_k^3 k_l^4 a_i (a_j g(a_k, a_l) - g(a_j a_k, a_l) + g(a_j, a_k a_l)).$$

Since N is of left dimension s over R, not more than s of the $g(a_i, a_j)$ can be left independent over R. If fewer than s of the values of g were left independent, it would then follow that N was of dimension less than s, contrary to hypothesis. Hence, some set of s values of g must be left linearly independent over R, and so form a left basis of N over R.

Using this basis of N over R, we can then prove the following theorem.

Theorem 5. Let R be a division algebra over K, and let $A = ext(R,N,g)$ where $N = Rg(R,R)R$. Then N is in the center of A if and only if each projection of g with respect to $\{g_i\} = \{g(b_i,c_i)\}$ is a 2-dimensional cocycle of $Z^2(R,R)$.

Proof: In the case where R was a semi-simple algebra over K, it was noticed that $N = \sum N_i = \sum Rn_i$, where the N_i are irreducible R-left spaces. Since R is a division algebra, if n is in N, then $n = \sum r_i n_i$ for r_i in R, where now the r_i are unique for a given n. It has already been demonstrated that a set $\{g(b_i,c_i)\}$ of values of g is left independent over R and spans N. Consequently, if g_i denotes $g(b_i,c_i)$, Rg_i is an irreducible R-left module of N. Therefore, N is the direct sum of its irreducible R-left spaces, that is, $N = \sum Rg_i$, hence for any element n of N,

$n = \sum r_i g_i$ for a unique r_i in R. In particular, for every x and y in R, $g(x,y) = \sum \lambda_i(x,y)g_i$. It may then be observed that $\lambda_i(b_j,c_j) = 0$ for $i \neq j$, $\lambda_i(b_i,c_i) = 1$ for all i. Since A is associative, g is a 2-dimensional cocycle. Using this information, there follows

$$(5{:}1)\quad 0 = \sum_j (b_i\lambda_j(c_i,z)g_j - \lambda_j(b_ic_i,z)g_j + \lambda_j(b_i,c_iz)g_j - \lambda_j(b_i,c_i)g_jz).$$

Equation (5:1) then becomes

$$(5{:}2)\qquad \sum_j (\delta\lambda_j)(b_i,c_i,z)g_j + zg_i - g_iz = 0,$$

for all i. If each λ_j is a cocycle, equation (5:2) implies $zg_i = g_iz$ for every z in R and every i. Hence, if λ_j is a cocycle for each j, then N is in the center of A.

To show the converse, make use of the vanishing of the coboundary of g:

$$(5{:}3)\quad 0 = \sum(x\lambda_j(y,z)g_j - \lambda_j(xy,z)g_j + \lambda_j(x,yz)g_j - \lambda_j(x,y)g_jz).$$

Since N is central, it follows immediately that λ_i is a cocycle for each i.

Corollary. If N is in the center of A, each projection of g with respect to any left basis of N over R is a cocycle.

Proof: This statement is implied by equation (5:3), which did not depend on any specific basis for N over R.

It may be pointed out here that the normality of g over K implies the normality of the projections of g and conversely.

Lemma 8. Under the same hypothesis as Theorem 5, with N in the center of A, the projections of g on R with respect to any left basis $\{n_i\}$ of N are cobounding cocycles if and only if g is a cobounding cocycle.

Proof: Suppose that g is a cobounding cocycle. Then there exists a function f, such that
$g(x,y) = xf(y) - f(xy) + f(x)y$. But since f(x) is in N, then $f(x) = \sum h_i(x)g_i$, $h_i(x)$ in R, for all x in R. Consequently,
$g(x,y) = x\sum h_i(y)g_i - \sum h_i(xy)g_i + \sum h_i(x)g_i y = \sum \lambda_i(x,y)g_i$.
But then equating coefficients of the g_i one obtains
$\lambda_i(x,y) = (\delta h_i)(x,y)$, for each i. The converse is obviously true.

Actually, the m.u. characteristics of an algebra are in many cases determined by the center of the algebra. For instance: The direct product $D_1 \times D_2$ of two division algebras over a field K is m.u. if and only if the direct product $Z_1 \times Z_2$ of the centers Z_1 of D_1 and Z_2 of D_2 is m.u. (26). Furthermore, one can prove the following theorem:

Theorem 6. If $A = \mathrm{ext}(R,N,g)$ is a singular extension of a division algebra R over K, and $A' = \mathrm{ext}(Z,N,g)$, where Z is the center of R, then g is a non-cobounding cocycle over R if and only if g is a non-cobounding cocycle over Z.

Proof: If g is non-cobounding over Z, then there is no function f in $C^1(Z,N)$ such that $g = \delta f$ for all x and y in Z. But since Z is in R, the same may be said for all f, x, and y in R, so that g is non-cobounding over R.

Conversely, suppose g is non-cobounding over R, and that g is cobounding over Z. Then $A' = \mathrm{ext}(Z,N,g)$ is cleft, by Lemmas 1 and 7. As a result, there exists a set of representatives Z^* for A'/N such that $Z^* = Z$ and Z^* is contained in A', hence, in A. But then A/N is a semi-simple algebra over Z^* which is isomorphic over K to Z, so that A/N is central. But every semi-simple central algebra is separable, hence A/N is separable, so that A is cleft (1). Hence, g is a coboundary. But this contradicts the hypothesis that g was non-cobounding, and the theorem is proved.

To apply the results of this chapter to associative extensions A, it is sufficient to assume the singularity of A. Furthermore, it is known (12) that if N is a vector space over a division ring R, such that rn = nr for all n in N and r in R (that is, trivially two-sided), then R is a field. In view of Theorem 5, a central radical will be assumed, so that R will be an algebraic extension field of K, henceforth. Thus A is reduced to a singular, completely primary algebra with central radical N and residue system an algebraic extension field of finite degree over K modulo N.

III. PROJECTIONS AND MAXIMAL UNCLEAVAGE

A. Maximally Uncleft Singular Extensions of Algebraic Extension Fields

Theorem 7. If $A = \text{ext}(R,N,g)$ with N central, is a singular extension by R, then A is m.u. if and only if all the projections of g with respect to bases of N over R are non-cobounding cocycles.

Proof: It follows as in Chapter II that $g(x,y) = \sum \lambda_i(x,y)n_i$ where the λ_i are unique for a given basis $\{n_i\}$ of N over R. If A is a singular m.u. extension by R, then by Theorem 1 g is a non-cobounding cocycle. Suppose that $\lambda_j(x,y)$ were cobounding, for some j. Then form

$$(7:1) \qquad g(x,y) - \lambda_j(x,y)n_j = \sum_{i \neq j}^{s} \lambda_i(x,y)n_i .$$

But $\lambda_j(x,y)n_j$ is cobounding since $\lambda_j(x,y)$ is. Hence, by Lemma 1, equation (7:1) defines a non-cobounding cocycle $g'(x,y)$ corresponding to a different linear set of representatives $\{A'_x\}$ of A/N. By hypothesis, A is m.u., so that this new set of representatives $\{A'_x\}$ generates A, and in particular, N. But N is of dimension s over R, while equation (7:1) shows that N is of dimension (s - 1) over R. This provides the necessary contradiction, so that all the projections of g on R are non-cobounding cocycles.

For the converse, suppose that all projections of g with respect to bases of N over R are non-cobounding cocycles. Then using the fact (Lemma 7) that g is cobounding if and only if all the projections of g are cobounding, one concludes that g must be non-cobounding. Hence A is m.u. This concludes the proof.

The objection may arise that the argument for the necessary condition of Theorem 7 would also hold true even if all the λ_i were non-cobounding. This is, of course, not true. Suppose g is non-cobounding, A is m.u., and all the λ_i are non-cobounding. Then $g(x,y) - \lambda_j(x,y)n_j = \sum_{i \neq j}^{s} \lambda_i(x,y)n_i$. But $g(x,y) - \lambda_j(x,y)n_j$ is a difference of two non-cobounding cocycles of R to N, and must be a coboundary, by Lemma 1. $\sum_{i \neq j}^{s} \lambda_i(x,y)n_i$ is therefore a sum of non-cobounding cocycles which is a coboundary. But a sum of cocycles is cobounding if and only if each term in the smm is cobounding, as was shown in Lemma 8. It follows then that $\sum_{i\neq j}^{s} \lambda_i(x,y)n_i$ does not generate a sub-algebra of N.

It is true that for each m.u. singular extension A and each non-cobounding 2-cocycle of R into N there exists a set of projections $\{\lambda_i\}$ each of which is non-cobounding. Is the converse of this statement true? The answer constitutes the next theorem.

Theorem 8. Given a set of s linearly independent (over R) non-cobounding 2-cocycles $\{\lambda_i\}$ of $H^2(R,R)$, then there exists a singular m.u. extension A by R which is determined by a non-cobounding cocycle g having the set $\{\lambda_i\}$ as its projections in R.

Proof: Let N' be a trivially 2-sided vector space over R of dimension t over R, where $t \geq s$. Choose s elements n_i of N' which are linearly independent over R. Define

$$(8:1) \qquad g(x,y) = \sum_{1}^{s} \lambda_i(x,y)n_i .$$

It is first of all true that g is a non-cobounding 2-cocycle of R into N'. For if g were cobounding, then there would exist a 1-cochain f of $C^1(R,N')$ such that $g(x,y) = (\delta f)(x,y)$ for all x and y in R. But

(8:2) $$f(x) = \sum_{1}^{t} f_i(x)n_i$$

where $f_i(x)$ is in R for all i. This implies that

(8:3) $$g(x,y) = \sum_{1}^{t} (\delta f_i)(x,y)n_i.$$

But equations (8:3) and (8:1) imply

(8:4) $$\lambda_i(x,y) = (\delta f_i)(x,y)$$

for i = 1,---,s, contrary to the supposition that the projections λ_i were non-cobounding. The 2-cocycle g is, therefore, non-cobounding.

Next, suppose that for all x and y in R, g(x,y) could be written as a combination of fewer than s elements from N', say as

(8:5) $$g(x,y) = \sum_{1}^{s'} u_i(x,y)n_i', \quad s' < s.$$

Since

(8:6) $$n_i' = \sum_{1}^{s} d_{ij}n_j$$

for d_{ij} in R, it follows from equation (8:5) that

(8:7) $$g(x,y) = \sum_{1}^{s'} u_i(x,y)\sum_{j}^{s} d_{ij}n_j.$$

Now if one uses equations (8:1) and (8:7), it follows that

(8:8) $$\lambda_j(x,y) = \sum_{1}^{s'} d_{ij}u_i(x,y)$$

for all j, where the sum is on i. But the equations in (8:8) imply that the $\{\lambda_i\}$ are not independent over R, since each

λ_i can be written as a combination of fewer than s cocycles of $Z^2(R,R)$. Therefore, $g(x,y)$ can be written as a linear combination of no fewer than s elements from N', so that if we define $N'^2 = 0$, then $A = ext(R,N,g)$, where $N = Rg(R,R)R$, is an uncleft extension. Suppose $g'(x,y)$ is a new cocycle formed by using N to modify A_x. Then

(8:9) $g'(x,y) - g(x,y) = (\delta f)(x,y) = \sum(\lambda_i'(x,y) - \lambda_i(x,y))n_i$,

where the λ_i' are non-cobounding. Then as in the previous part of the proof, $\lambda_i' - \lambda_i = \delta f_i$ for each i. Therefore, λ_i' is cohomologous to λ_i for each i, so that the λ_i' are also independent. Hence, $Rg'(R,R)R = N$. This completes the proof.

An unexpected side light was uncovered in the proof of this theorem: If g and g' are two distinct non-cobounding 2-cocycles of $H^2(R,N)$, which are linearly dependent over a base field, then g and g' are not cohomologous. Thus, any two distinct dependent non-cobounding cocycles determine non-isomorphic singular extension algebras.

B. An Example

It should be pointed out that an algebra with a central radical need not be a commutative algebra. For example, let K be the field $P_2(\alpha_1, \alpha_2)$ of characteristic 2, obtained by adjoining the algebraically independent indeterminates α_1 and α_2 to the prime field of integers, P_2. Let R be the field $K(a_1, a_2)$ obtained by adjoining a_1 and a_2 to K, where the minimum polynomials in K of a_1 and a_2 are $x^2 - \alpha_1$ and $x^2 - \alpha_2$, respectively. R, then, is a pure inseparable field, and being a field, is semi-simple. Construct an algebra A as follows: Make the correspondence $1 \leftrightarrow A_1$, $a_1 \leftrightarrow A_{a_1}$, $a_2 \leftrightarrow A_{a_2}$, $a_1a_2 \leftrightarrow A_{a_1a_2}$, for 1, a_1, a_2, and a_1a_2 basis elements of R over K, where A_1, A_{a_1}, A_{a_2}, and $A_{a_1a_2}$ will be a linear set of representatives of the residue classes of A/N to which 1, a_1, a_2, a_1a_2 correspond, respectively. Then define a 2-cochain g as follows: Let g be normalized over K, and let $g(x,y) = A_xA_y - A_{xy}$, where, $g(a_1,a_1) = 0$, $g(a_2,a_2) = 0$, $g(a_1,a_2) = 0$, and $g(a_2,a_1) = n$ in N. For convenience, adopt the notation $A_1 = 1$, $A_{a_1} = a_1$, and so forth. Then define multiplication for the singular A = ext(R,N,g) with central radical by:

1	a_1	a_2	a_1a_2	n	a_1n	a_2n	a_1a_2n
a_1	α_1	a_1a_2	α_1a_2	a_1n	α_1n	a_1a_2n	α_1a_2n
a_2	a_1a_2+n	α_2	$\alpha_2a_1+a_2n$	a_2n	a_1a_2n	α_2n	α_2a_1n
a_1a_2	$\alpha_1a_2+a_1n$	α_2a_1	$\alpha_1\alpha_2+a_1a_2n$	a_1a_2n	α_1a_2n	α_2a_1n	$\alpha_1\alpha_2n.$

It can be verified that this is an associative as well as a non-commutative multiplication, thus guaranteeing that g is a cocycle. Furthermore, if the residue system were changed to $1 + n_1$, $a_1 + n_2$, $a_2 + n_3$, $a_1a_2 + n_4$, for n_i in N, then $(1 + n_1)^2 = 1$, $(a_1 + n_2)^2 = \alpha_1$, $(a_2 + n_3)^2 = \alpha_2$, and $(a_1a_2 + n_4)^2 = \alpha_1\alpha_2 + a_1a_2n$. Consequently, this new residue system generates the whole radical N. Suitable linear combinations of this basis of N over R would then produce n_1, n_2, n_3, and n_4. As a result, 1, a_1, a_2, a_1a_2 would be recovered from the new residue system. The conclusion is that every residue set of A/N generates A, or that A is m.u. This is an example, therefore, of an m.u., non-commutative, singular algebra with a central radical.

In view of the reduction of the main problem to pure inseparable fields, as discussed in Section 5, Chapter I, the investigation of m.u. extensions will be continued in

Chapter IV with the restriction that R be a pure inseparable field extension of K.

IV. $H^2(R,R)$ AND M.U. EXTENSIONS

In this chapter m.u. extensions L^oM by a field R which is the direct product of simple pure inseparable field extensions will be analyzed. Since not all direct products are again fields, the first section of this chapter is devoted to a short discussion of necessary and sufficient conditions for a direct product to be a field.

A. Direct Products of Simple Pure Inseparable Field Extensions

The next theorem, and some considerations in the remaining part of this thesis depend heavily on a theorem of G. Pickert (18), which will be quoted below.

Pickert Theorem 1: For every m generators of a pure inseparable field extension L of K, of multiplicity m, the generators $a_1, a_2, ---, a_m$ can be so rewritten that

1. $a_i^{q_i} = \alpha_i \epsilon K(a_1^{q_i}, ---, a_{i-1}^{q_i})$, $q_i = p^{e_i}$, $e_i > 0$,
2. $\alpha_i^{1/p}$ is not in $K(a_1, ---, a_{i-1})$,
3. $e_1 \geqslant e_2 \geqslant --- \geqslant e_m$.

The representation $K(a_1,---,a_m)$ as given in this theorem is called special. Making use of this theorem, the next theorem can be proved.

Theorem 9. A pure inseparable field $K(a_1,a_2)$ of multiplicity two is a direct product of two simple pure inseparable fields $K(b_1)$ and $K(b_2)$ if and only if the degree of a_2 over K and over $K(a_1^{q_2})$ is the same.

Proof: Suppose $K(a_1,a_2) = K(b_1) \times K(b_2)$. Then by Pickert Theorem 1, $a_1^{q_1} = \alpha_1$ in K and $\alpha_1^{1/p}$ is not in K; $a_2^{q_2} = \alpha_2$ in $K(a_1^{q_2})$, and $\alpha_2^{1/p}$ is not in $K(a_1)$. Consequently, $x^{q_2} - \alpha_2$ is irreducible over $K(a_1)$, so that the degree of a_2 over $K(a_1)$ is also q_2. The degree of $K(a_1,a_2)$ over K is then $q_1 q_2$.

Since $K(b_1) \times K(b_2)$ is a pure inseparable field with generators b_1 and b_2 over K, these generators can be reordered so that the degree of b_1 over K is q_1, while the degree of b_2 over $K(b_1^{q_2})$ is q_2. Further, let the degree of b_2 over K be $q_2' \geqslant q_2$. Then $q_1 q_2 = q_1 q_2'$, so that $e_2' = e_2$.

Consequently, the degree of a_2 over $K(a_1^{q_2})$ is the same as the degree of b_2 over K, so that the degree of a_2 over

$K(a_1^{q_2})$ is the same as the degree of a_2 over K.

Conversely, let $L = K(b_1) \times K(b_2)$ be the direct product of two simple pure inseparable fields. Then L contains $K(b_1) \times 1$, which is a field of degree q_1 over K. But b_2 is of degree q_2 over $K(b_1) \times 1$. It follows that $(K(b_1) \times 1)(b_2)$ is a field of degree $q_1 q_2$ contained in L. Since L is of degree $q_1 q_2$ over K, then $L = (K(b_1) \times 1)(b_2)$, and is a field, $K(a_1, a_2)$.

Corollary. If $K(a_1) \times K(a_2)$ is a field, where $K(a_1)$ and $K(a_2)$ are pure inseparable field extensions of K of characteristic p, then $K(a_1) \times K(a_2)$ is of multiplicity two.

It is noted here that Theorem 9 can be generalized:

Theorem 10. If $L = K(a_1, ---, a_n)$ is a pure inseparable field over K of multiplicity n, then L is a direct product of simple pure inseparable fields if and only if the degree of a_i over $K(a_1^{q_i}, ---, a_{i-1}^{q_i})$ is the same as the degree of a_i over K.

The proof of this theorem closely resembles the proof of Theorem 9 and will be omitted.

Definition 14. (25) Two fields L and M over a common field K are called disjoint over K if and only if K is the only sub-field of L isomorphic to a sub-field of M.

Theorem 11. $K(a_1) \times K(a_2)$ is a field if and only if $K(a_1)$ and $K(a_2)$ are disjoint over K.

Proof: It has been proved by Vinograde (25) that there exists a unique maximum sub-field of $K(a_1)$ isomorphic to a sub-field of $K(a_2)$. Suppose U is a sub-field of $K(a_1)$ and V is a sub-field of $K(a_2)$ such that $K \leq U \leq K(a_1)$ and $K \leq V \leq K(a_2)$, with $U \cong V$. In the direct product both U and V are sub-fields. Then $u - v$ is a nilpotent element of the direct product, for u in U and v the corresponding element in V. Consequently, if $K(a_1)$ and $K(a_2)$ have isomorphic sub-fields containing K, then $K(a_1) \times K(a_2)$ is not a field.

Conversely, if $K(a_1)$ and $K(a_2)$ have maximum isomorphic sub-field K, then, according to Pickert (18) there is no nilpotent element in the direct product. It follows, therefore, that the direct product is a field.

As an illustration, let $K = P_p(\alpha_1, \alpha_2)$, where P_p is the field of integers modulo the prime integer p, and α_1 and α_2

are algebraically independent indeterminates over P_p. Let the minimum polynomials of a_i over K be $x_i^{q_i} - \alpha_i$ where $q_i = p^{e_i}$, for some positive integers e_i, for $i = 1,2$. Form $K(a_1)$ and $K(a_2)$. Then $K(a_1) \neq K(a_2)$, for if $K(a_1) = K(a_2)$, then a_1 corresponds to some polynomial $h(a_2)$ in a_2 over K, that is, $a_1 \leftrightarrow h(a_2)$. Then $a_1^{q_1q_2} = \alpha_1^{q_2} \leftrightarrow h(a_2)^{q_1q_2} = \alpha_1^{q_2}$. But $h(a_2)^{q_1q_2} = h'(a_2^{q_1q_2}) = \alpha_1^{q_2}$ for some polynomial h' in K. Consequently, $h'(\alpha_2^{q_1}) = \alpha_1^{q_2}$. But this is a polynomial in α_1 and α_2 with coefficients in $P_p(\alpha_1,\alpha_2)$. This, then, is a polynomial in α_1 and α_2 over P_p, contrary to the hypothesis of algebraic independence over P_p. It follows then that $K(a_1)$ and $K(a_2)$ are non-isomorphic relative to K.

It will now be shown that these fields are disjoint over K. The only sub-fields of $K(a_1)$ are $K(a_1) \supset \text{---} \supset K(a_1^{p^s}) \supset \text{---} \supset K$, while for $K(a_2)$, $K(a_2) \supset \text{---} \supset K(a_2^{p^s}) \supset \text{---} \supset K$. Suppose that $K(a_1^{p^s}) = K(a_2^{p^s})$. Then $(a_1^{p^s})^{p^{e_1-s}} = \alpha_1$ and $(a_2^{p^s})^{p^{e_2-s}} = \alpha_2$. In the same way that $K(a_1)$ and $K(a_2)$ were shown to be non-isomorphic over K, now $K(a_1^{p^s})$ and $K(a_2^{p^s})$ can be shown to be non-isomorphic. It follows that $K(a_1)$ and $K(a_2)$ have no isomorphic sub-fields other than K, so that by Theorem 11, the direct product is a field.

B. Non-cobounding 2-cocycles on Simple Pure Inseparable Fields

The next problem is to determine the structure of 2-cocycles on simple pure inseparable fields with a view to discovering the structure of 2-cocycles on direct products of pure inseparable fields. In this section, K(a) is assumed to be a simple pure inseparable field extension of K. Let the minimum equation of a over K be $x^q - \alpha = 0$, where $q = p^e$ for some integer e.

Theorem 12. There is a one-to-one correspondence between the m.u. singular extensions A by K(a) with 1-dimensional radical and the non-cobounding 2-cocycles of K(a) into K(a).

Proof: Let N be the radical of A. Since N is 1-dimensional, $g(x,y) = \lambda(x,y)g'$ for every x and y in K(a) and for some value g' of g(x,y). It is certainly true that g is a non-cobounding 2-cocycle, so that λ is also a non-cobounding

2-cocycle, by Theorem 7. Hence, each non-cobounding 2-cocycle on K(a) to K(a) determines exactly one non-cobounding 2-cocycle g on K(a) to N, and conversely. By Hochschild's theorem (7, p. 67), one has, in particular, that there is a one-to-one correspondence between the non-cobounding 2-cocycles of K(a) into N and singular m.u. field extensions. Hence, there is a one-to-one correspondence between non-cobounding 2-cocycles of K(a) into K(a) and m.u. field extensions.

For the simple pure inseparable field extension $R = K(a)$, one can define a linear set of representatives for A/N by $a^i \leftrightarrow A_{a^i}$, for a^i in R and A_{a^i} in A. Define multiplication in the residue system by $A_a^i = A_{a^i}$ for $i < q$, and $A_a^q = A_{a^q} + n$, where n is a non-zero radical element. Under these circumstances, A is m.u. For if the residue set is replaced by $A_{a^i} + m_i$, for m_i in N, then all of N is generated by the new residue set, since $(A_a + m_1)^q = A_a^q = \alpha + n$, where $\alpha = A_{a^q}$. Now it is possible to specify the exact form of each non-cobounding 2-cocycle of R to R.

Theorem 13. Let A be an m.u. singular extension by R, with a 1-dimensional radical N. Let g be a non-cobounding 2-cocycle of R into N. Then every 2-cocycle of R into R is non-cobounding and has the form $\lambda(x,y) = \sum_{j=1}^{q-1} \sum_{i=q-j}^{q-1} x_j y_i a^{i+j-q}$, uniquely within multiples from R.

Proof: Make a choice of residue system $a^i \leftrightarrow A_{a^i}$ for a^i in R and A_{a^i} in A. Define g as follows:

$$\text{(13:1)} \qquad \begin{aligned} g(a^i,a^j) &= 0 \text{ for } i + j < q, \\ g(a^i,a^j) &= a^{i+j-q}n \text{ for } i + j \geqslant q. \end{aligned}$$

This implies that $A_{a^i} = A_a^i$ for $i < q$, and $A_{a^q} = A_a^q + n = \alpha + n$, α in K, n in N. Then for all x and y in R,

$$\text{(13:2)} \qquad \begin{aligned} g(x,y) &= g\left(\sum_0^{q-1} x_j a^j, \sum_0^{q-1} y_i a^i\right) \\ &= \sum_{j=1}^{q-1} \sum_{i=q-j}^{q-1} x_j y_i a^{i+j-q} n. \end{aligned}$$

It follows then that the general cocycle λ on R into R is

$$\text{(13:3)} \qquad \lambda(x,y) = \sum_{j=1}^{q-1} \sum_{i=q-j}^{q-1} x_j y_i a^{i+j-q}.$$

λ is non-cobounding since g is non-cobounding. It may be stated, in fact, that any 2-cocycle g of R into N of the form given in equation (13:2) is non-cobounding. For suppose g were cobounding on R. Then for some linear map f of R into N,

$$g(x,y) = xf(y) - f(xy) + f(x)y, \tag{13:4}$$

for all x and y in N. In particular,

$$g(a^i,a^{q-1}) = a^i f(a^{q-1}) - \alpha f(a^{i-1}) + a^{q-1}f(a^i), \tag{13:5}$$

for $i = 1,---,q-1$. Consequently,

$$\begin{aligned} &\sum_{i=1}^{q-1} a^{q-i-1}g(a^i,a^{q-1}) \\ &= \sum(a^{q-1}f(a^{q-1}) - \alpha a^{q-i-1}f(a^{i-1}) + a^{2q-i-2}f(a^i)) \\ &= qa^{q-1}f(a^{q-1}) \\ &= 0. \end{aligned} \tag{13:6}$$

Using the fact that $g(a^i,a^{q-1}) = a^{i-1}n$ for $i = 1,---,q-1$, it follows that

$$\sum_{i=1}^{q-1} a^{q-i-1}a^{i-1}n = \sum_{i=1}^{q-1} a^{q-2}n = (q-1)a^{q-2}n = 0. \tag{13:7}$$

But $(q-1)$ does not vanish, since if $p = 2$, $(q-1)$ is odd and is not a power of p. If $p \neq 2$, $(q-1)$ is even and is not a power of p. Thus, on the assumption that g is cobounding, a contradiction is introduced. Hence, the only 2-cocycles on R into a 1-dimensional radical N of an m.u. algebra A are non-cobounding. But then all 2-cocycles of R into R are non-cobounding. Since there is but one independent in $H^2(R,R)$, and since, by Theorem 8, to each set of s independent λ's there corresponds an m.u. singular extension, we conclude that:

Theorem 14. Every m.u. singular extension A of a simple pure inseparable field K(a) has a radical N of dimension one over K(a), and each A is determined by a scalar multiple u(A) of λ.

We are now equipped to direct attention to L^oM in the case that modulo its radical L^oM is a direct product of simple pure inseparable fields.

C. 2-cocycles on Pure Inseparable Fields of Multiplicity Two Which Are Direct Products of Simple Pure Inseparable Fields

It will be assumed in this section that $K(a_1) \times K(a_2) = R$ is a pure inseparable field. It will be shown that the study of non-cobounding 2-cocycles on R into R in this case is, in a sense, degenerate. That is, the value of λ in $H^2(R,R)$ is determined by its values on $K(a_1) \times 1$ and $1 \times K(a_2)$.

Theorem 15. Let λ belong to $Z^2(R,R)$. Then λ is determined by its values on $K(a_1) \times 1$ and $1 \times K(a_2)$.

Proof: Consider the additive group R' of R as an R module. Form the commutative extension $A = \text{ext}(R, R', \lambda)$ where A contains R' so that $A/R' = R$ and $R' = R\lambda(R,R)R$. Define multiplication in R' by $R'^2 = 0$. Choose a residue system in A by $r \leftrightarrow A_r$ for r in R and A_r in A. A basis for R is $\{a_1^i a_2^j\}$, $i = 0,1,---,q_1-1$, $j = 0,1,---,q_2-1$, where $q_i = p^{e_i}$, p being the characteristic of K, and e_1 and e_2 being integers. Define multiplication of the residue system by $A_r A_s - A_{rs} = \lambda(r,s)$, where

$$(15:1)\quad \begin{aligned} &\lambda(a_1^i, a_1^j) = 0 \text{ for } i + j < q_1, \\ &\lambda(a_2^i, a_2^j) = 0 \text{ for } i + j < q_2, \\ &\lambda(a_1^i, a_1^j) = a_1^{i+j-q_1} r_1 \text{ for } i + j \geqslant q_1 \text{ and } r_1 \text{ in } R', \\ &\lambda(a_2^i, a_2^j) = a_2^{i+j-q_2} r_2 \text{ for } i + j \geqslant q_2 \text{ and } r_2 \text{ in } R'. \end{aligned}$$

From these definitions, it follows that

$$(15:2)\quad \begin{aligned} \lambda(a_1^i a_2^j, a_1^k a_2^l) &= 0 \text{ for } i + k < q_1 \text{ and } j + l < q_2, \\ &= a_1^{i+k-q_1} a_2^{j+l} r_1 \text{ for } i + k \geqslant q_1 \\ &\qquad \text{and } j + l < q_2, \\ &= a_2^{j+l-q_2} a_1^{i+k} r_2, \text{ for } i + k < q_1 \\ &\qquad \text{and } j + l \geqslant q_2, \\ &= a_1^{i+k-q_1} a_2^{j+l-q_2} (\alpha_1 r_2 + \alpha_2 r_1) \\ &\qquad \text{for } i + k \geqslant q_1 \text{ and } j + l \geqslant q_2. \end{aligned}$$

The value of λ is completely determined, therefore, by its values on $K(a_1) \times 1$ and $1 \times K(a_2)$.

If λ is a non-cobounding 2-cocycle of R to R', λ is even more specifically determined by its values on $K(a_1) \times 1$ and $1 \times K(a_2)$. To show this, let $\lambda^1(x,y)$ be the value of λ when x and y belong to $K(a_1) \times 1$, and let $\lambda^2(x,y)$ be the value of λ when x and y come from $1 \times K(a_2)$. Considering $A^1 = \mathrm{ext}(K(a_1) \times 1, R', \lambda^1)$, by Theorem 14, since $K(a_1) \times 1$ is a pure inseparable field extension, then A^1 has a 1-dimensional radical and λ^1 is unique. Similarly for λ^2.

Theorem 16. λ in $Z^2(R,R')$ is a non-cobounding 2-cocycle if and only if λ^1 or λ^2 is non-cobounding.

Proof: λ can be written as

(16:1) $$\lambda(x,y) = \sum r_i^1(x,y)\lambda_i^1 + \sum r_i^2(x,y)\lambda_i^2,$$

where r_i^1 and r_i^2 map R into R' and λ_i^1 is λ^1 evaluated at specific pairs (a_1^i, a_1^j) of $K(a_1) \times 1$, while λ_i^2 is λ^2 evaluated at specific pairs (a_2^i, a_2^j) of $1 \times K(a_2)$.

Suppose λ^1 or λ^2 is non-cobounding. Then λ is not cobounding for all values of R, so that λ is a non-cobounding cocycle.

Next, suppose λ^1 and λ^2 were cobounding. Then if x and y are any elements of $K(a_1) \times 1$,

(16:2) $$\lambda(x,y) = \lambda^1(x,y) = \sum r_i^1(x,y)\lambda_i^1,$$

so that $\sum r_i^1(x,y)\lambda_i^1$ is cobounding. Similarly, $\sum r_i^2(x,y)\lambda_i^2$ is cobounding. Since the value of λ is determined by its values on the factors of the direct product, then λ is cobounding for all x and y in R, and the theorem is proved.

This last theorem permits the study of m.u. singular extension algebras of pure inseparable fields which are direct products to be restricted to the study of the separate factors of the direct product. The important question now arises: How many non-cobounding cocycles on R to R' exist? This is of course equivalent to asking how many non-isomorphic singular extensions there are of R by R'. The answer is:

Theorem 17. For $R = K(a_1) \times K(a_2)$, a pure inseparable field extension of multiplicity two, there are three linearly independent non-cohomologous non-cobounding cocycles. Consequently, there are three possible classes of m.u. singular extension algebras.

Proof: Since $\lambda(x,y) = \sum r_i^1\lambda_i^1 + \sum r_i^2\lambda_i^2$ is non-cobounding if and only if λ^1 or λ^2 is non-cobounding, there are three cases to be considered:

1. λ^1 cobounding, λ^2 non-cobounding,
2. λ^1 non-cobounding, λ^2 cobounding,
3. λ^1 and λ^2 non-cobounding.

Corresponding to each case will be a distinct λ. From the first case arises only one distinct λ. For if λ^1 is a cobounding cocycle, and if λ^{1*} is a different cobounding cocycle, then let

$$
\begin{aligned}
\lambda(x,y) &= \sum r_i^1\lambda_i^1 + \sum r_i^2\lambda_i^2 \\
\lambda^*(x,y) &= \sum r_i^1\lambda_i^{1*} + \sum r_i^2\lambda_i^2 \qquad (17:1)\\
\lambda(x,y) - \lambda^*(x,y) &= \sum r_i^1(\lambda_i^1 - \lambda_i^{1*}).
\end{aligned}
$$

But $\lambda - \lambda^*$ is a coboundary. Hence λ^1 and λ^{1*} are cohomologous. It follows then, that from each of case 1 and 2, there can be obtained only one non-cobounding cocycle, λ. For the third case, since there is but one non-cobounding cocycle in $K(a_1) \times 1$ and in $1 \times K(a_2)$, then only one more non-cobounding λ is produced. Thus, a total of at most three non-cobounding, independent cocycles λ in $H^2(R,R')$

can be found.

An interesting fact about normalized non-cobounding cocycles of R into R' is the following: If the characteristic of K is 2, and if the minimum equations satisfied by a_1 and a_2 on K are $x^2 - \alpha_1 = 0$, and $x^2 - \alpha_2 = 0$, then there are no normalized cobounding 2-cocycles on R to R', while for all other cases, there are cobounding 2-cocycles. Consider the case mentioned. Let λ be an element of $Z^2(R,R')$. A basis of R over K is 1, a_1, a_2, a_1a_2. Then since we are considering normalized cocycles, the only values of λ which do not vanish on either factor of the direct product are $\lambda_1^1 = \lambda(a_1,a_1)$ and $\lambda_1^2 = \lambda(a_2,a_2)$. From this follows $\lambda(a_1,a_1a_2) = \lambda_1^1 a_2$, $\lambda(a_2,a_1a_2) = \lambda_1^2 a_1$, and $\lambda(a_1a_2,a_1a_2) = \alpha_1\lambda_1^2 + \alpha_2\lambda_1^1$. For all x in R, $x = x_0 + x_1a_1 + x_2a_2 + x_3a_1a_2$, where the x_i belong to K. It follows then that

$$\lambda(x,y) = (x_1y_1 + a_2(x_1y_3 + x_3y_1) + \alpha_2x_3y_3)\lambda_1^1$$
$$+ (x_2y_2 + a_1(x_2y_3 + x_3y_2) + \alpha_1x_3y_3)\lambda_1^2.$$

If D_1x denotes the derivation of x over $K(a_2) \times 1$ and D_2x denotes the derivation of x over $K(a_1) \times 1$, it is then true that

$\lambda(x,y) = (D_1xD_1y)\lambda_1^1 + (D_2xD_2y)\lambda_1^2$. If now D_1xD_1y are D_2xD_2y are cobounding, λ will be cobounding, and conversely. We will, therefore, examine $DxDy$ in general, that is, for characteristic p and exponent e. Let us suppose $DxDy$ is cobounding. Then there is some linear map f of R to R' such that $DxDy = xf(y) - f(xy) + f(x)y$. Evaluating $DxDy$ for certain basis elements, one obtains:

$$\begin{aligned} DaDa &= 1 = 2af(a) - f(a^2) \\ DaDa^2 &= 2a = af(a^2) - f(a^3) + a^2f(a) \\ &\ \vdots \\ DaDa^{q-1} &= (q-1)a^{q-2} = af(a^{q-1}) + a^{q-1}f(a). \end{aligned}$$

It follows from the form of the right hand side of these equations that $\sum_{i=1}^{q-1} a^{q-i-1}Da^iDa^{q-1} = 0$. But replacing Da^iDa^{q-1} by the value $ia^{i-1}(q-1)a^{q-2}$, this sum becomes $0 = \sum_{i=1}^{q-1} ia^{2q-4}(q-1) = \frac{1}{2}(q-1)^2qa^{2q-4}$. If $q = 2$, we obtain a contradiction. Hence, $DxDy$ is not cobounding when $q = 2$. If q is not 2, suppose $q = 2^e$ for $e > 1$. Then the equation becomes $2^{e-1}(2^e - 1)^2a^{2^{e+1}-4} = 0$, which is permissible since the characteristic is 2. If p is not even, then q is an odd number, so that $\frac{1}{2}(q-1)^2$ is an integer.

Then $\frac{1}{2}(q-1)^2 qa^{2q-4} = 0$. It is true then that there are no normalized cobounding 2-cocycles if $q = 2$, but there are in all other cases.

To illustrate the preceding discussion, let us determine the projections of g on R for the algebra constructed in Section B of Chapter III. In that case, it may be recalled, $g(x,y) = \lambda(x,y)n$ for all x and y in R, where $n = g(a_2,a_1)$. A basis for R over K is 1, a_1, a_2, and $a_3 = a_1a_2$. For any x and y in R, $x = \sum_{i=0}^{3} x_i a_i$, $y = \sum_{i=0}^{3} y_i a_i$, where $a_0 = 1$, so that $g(x,y) = \sum x_i y_j g(a_i,a_j)$, or

$$g(x,y) = (1\ x_1\ x_2\ x_3)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & a_2 \\ 0 & a_1 & 0 & a_1a_2 \end{pmatrix}\begin{pmatrix} 1 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} n.$$

The coefficient of n is $\lambda(x,y)$, hence, $\lambda(x,y) = (D_2x)(D_1y)$. But for the case $q = 2$, this expression is a non-cobounding 2-cocycle.

The last subject which will be treated in this section is the dimension of N/N^2 for $A = \mathrm{ext}(R,N,g)$, where $R = K(a_1) \times K(a_2)$.

Theorem 18. The dimension of N/N^2 over R for $A = \text{ext}(R,N,g)$ an m.u. extension of R, is 1, 2, or 3.

Proof: Using Theorem 3, for any $g(x,y)$ in N,

(18:1) $$g(x,y) = \sum \lambda_i(x,y)g_i.$$

But by Theorem 17, there are exactly three possible independent λ's of R to R', say λ_1, λ_2, and λ_3. But then λ_i for $i > 3$ can be written as a linear combination of the λ_1, λ_2, and λ_3, say

(18:2) $$\lambda_i(x,y) = \sum r_{ij}\lambda_j(x,y)$$

for r_{ij} in R. Combining equations (18:1) and (18:2), there follows

(18:3) $$g(x,y) = \sum_i \sum_{j=1}^{3} r_{ij}\lambda_j(x,y)g_i,$$

or

(18:4) $$g(x,y) = \sum_{j=1}^{3} \lambda_j(x,y)\sum_i r_{ij}g_i.$$

Denoting $\sum r_{ij}g_i$ by n_j, this equation becomes

(18:5) $$g(x,y) = \sum_{j=1}^{3} \lambda_j(x,y)n_j.$$

Hence the dimension of N/N^2 is at most three over R.

D. The Case $L^oM = L \times M$

The m.u. character of the algebra $L^oM = L \times M$ when the direct product is not necessarily a field will be examined in this section. The results derived are from some general theorems proved by Pickert (18).

Pickert Theorem 2. For a finite pure inseparable field extension L of K and for any extension M of K, $L \times M$ is a completely primary algebra such that $L \times M/N \cong LM$ relative to M, where N is the radical of $L \times M$.

In the case which is considered here, both $L = K(a_1)$ and $M = K(a_2)$ are finite pure inseparable field extensions of K. Consequently, $L \times M$ is a completely primary algebra, and $L \times M/N \cong LM$ relative to L or to M.

Pickert Theorem 3. There is a special generation $L = K(a_1,---,a_m)$ such that $M(a_1,---,a_s)$ for $s \leq m$ is a special representation of LM over M.

Making use of both Pickert theorems, the following conclusions can be drawn for the case under consideration:

1. $L \times M/N \cong LM$ relative to M or to L.

2. LM over M can be written as LM = M, or LM = M(a_1).

3. LM over L can be written as LM = L, or LM = L(a_2).

It has been shown by Pickert (18), that if LM = M(a_1,---,a_s) in the general case, and if b_1,---,b_s are the elements of L X M/N corresponding to the a_1,---,a_s of LM, the rank of the radical generated by the b_1,---,b_s of L X M modulo N^2 is invariant under change of the residue set for L X M/N. In the case where s = m = 1, the radical of L X M modulo N^2 is generated by every residue set of L X M/N, that is, L X M is m.u. For if L X M/N = M(b_1) $\cong$ M(a_1), say, then the entire radical of L X M is generated by b_1, since there is no other element of the radical-residue-class field available for this generation. Consequently, under the invariance shown by Pickert, L X M is m.u. Under the light of this paragraph, the three cases listed above will be examined.

Suppose L X M/N $\cong$ LM relative to M. Then if the fact that the base field of LM is considered to be M is denoted by LM|M, it follows that L X M/N $\cong$ LM|M = M or M(a_1). If L X M/N $\cong$ M, then L X M is cleft over M since L X M contains M. If $L \times M/N \cong M(a_1)$, then by the discussion in the preceding paragraph, $L \times M$ is m.u. over M. Obviously, the same considerations will show that if $LM|L = L$, then $L \times M$ is cleft over L, while if $LM|L = L(a_2)$, then $L \times M$ is m.u. over L. Summarized, this says:

Theorem 19. If $L = K(a_1)$ and $M = K(a_2)$ are pure inseparable fields, then if

1. $LM|L = L$, then $L \times M$ is cleft over L.
2. $LM|L = L(a_2)$, then $L \times M$ is m.u. over L.
3. $LM|M = M$, then $L \times M$ is cleft over M.
4. $LM|M = M(a_1)$, then $L \times M$ is m.u. over M.

Corollary. If $L \times M$ is cleft over L or M, $L \times M$ is cleft over K. If $L \times M$ is m.u. over K, then $L \times M$ is m.u. over both L and M.

An algebra can cleave over one field but be uncleft over another field. Consider the following example: Let $L = K(a)$ with $a^{p^2} = \alpha$ in K, and $M = K(a^p)$. First, M is properly contained in L, so that $LM|M = M(a)$. But from Pickert Theorem 3, and the fact that $L(a^p) = L$, it is apparent that $LM|L = L$. In this case, $L \times M$ is m.u. with respect to M, but is cleft with respect to L.

V. BIBLIOGRAPHY

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